

# Phase transitions for Quantum Markov Chains associated with Ising type models on a Cayley tree

FARRUKH MUKHAMEDOV

*Department of Computational & Theoretical Sciences,  
Faculty of Science, International Islamic University Malaysia,  
P.O. Box, 141, 25710, Kuantan, Pahang, Malaysia  
E-mail: far75m@yandex.ru, farrukh\_m@iiium.edu.my*

ABDESSATAR BARHOUMI

*Department of Mathematics  
Nabeul Preparatory Engineering Institute  
Campus Universitaire - Mrezgua - 8000 Nabeul,  
Carthage University, Tunisia  
E-mail: abdessatar.barhoumi@ipein.rnu.tn*

ABDESSATAR SOUISSI

*Department of Mathematics,  
Marsa Preparatory Institute for Scientific and Technical Studies  
Carthage University, Tunisia  
E-mail: s.abdessatar@hotmail.fr*

## Abstract

The main aim of the present paper is to prove the existence of a phase transition in quantum Markov chain (QMC) scheme for the Ising type models on a Cayley tree. Note that this kind of models do not have one-dimensional analogous, i.e. the considered model persists only on trees. In this paper, we provide a more general construction of forward QMC. In that construction, a QMC is defined as a weak limit of finite volume states with boundary conditions, i.e. QMC depends on the boundary conditions. Our main result states the existence of a phase transition for the Ising model with competing interactions on a Cayley tree of order two. By the phase transition we mean the existence of two distinct QMC which are not quasi-equivalent and their supports do not overlap. We also study some algebraic property of the disordered phase of the model, which is a new phenomena even in a classical setting.

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## 1 Introduction

One of the basic open problems in quantum probability is the construction of a theory of quantum Markov fields, that are quantum processes with multi-dimensional index set. This program concerns the generalization of the theory of Markov fields (see [24],[32])) to a non-commutative setting, naturally arising in quantum statistical mechanics and quantum field theory.

The quantum analogues of Markov chains were first constructed in [1], where the notion of quantum Markov chain (QMC) on infinite tensor product algebras was introduced. The reader is referred to [15, 26, 34, 38, 44] and the references cited therein, for recent developments of the theory and the applications.

The main aim of the present paper is to prove the existence of phase transitions for a class of quantum Markov chains associated with Ising type models on a Cayley tree. This paper deals with two problems: The construction of quantum Markov fields on homogeneous (Cayley) trees; the existence of a phase transition for special models of such fields. Both problems are non-trivial and, to a large extent, open. In fact, even if several definitions of quantum Markov fields on trees (and more generally on graphs) have been proposed, a really satisfactory, general theory is still missing and physically interesting examples of such fields in dimension  $d \geq 2$  are very few. In this paper, from QMC perspective, we are going to establish the phase transition for the classical Ising model with competing interactions on a Cayley tree. Note that the phase transition notion is based on the quasi-equivalence of QMC which differs from the classical one (in the classical setting, to establish the phase transition for the considered model, it is sufficient to prove the existence of at least two different solutions of associated renormalized equations (see [31, 50])). Therefore, such a phase transition is purely noncommutative, and even for classical models, to check the existence of the phase transition is not a trivial problem (we point out that the quasi-equivalence of product states, which correspond to the classical models without interactions, was considered in [48]).

We notice that first attempts to construct a quantum analogue of classical Markov fields have been done in [3]-[6],[9, 33, 27, 36]. In these papers the notion of *quantum Markov state*, introduced in [8], extended to fields as a sub-class of the quantum Markov chains. In [7] a more general definition of quantum Markov states and chains, including all the presently known examples, have been extended. Note that in the mentioned papers quantum Markov fields were considered over multidimensional integer lattice  $\mathbb{Z}$ . This lattice has so-called amenability property. Moreover, analytical solutions (for example, critical temperature) does not exist on such lattice. But investigations of phase transitions of spin models on hierarchical lattices showed that there are exact calculations of various physical quantities (see for example, [19, 47]). Such studies on the hierarchical lattices begun with the development of the Migdal-Kadanoff renormalization group method where the lattices emerged as approximants of the ordinary crystal ones. On the other hand, the study of exactly solved models deserves some general interest in statistical mechanics [19]. Therefore, it is natural to investigate quantum Markov fields over hierarchical lattices. For example, a Cayley tree is the simplest hierarchical lattice with non-amenable graph structure [45]. This means that the ratio of the number of boundary sites to the number of interior sites of the Cayley tree tends to a nonzero constant in the thermodynamic limit of a large system. Nevertheless, the Cayley tree is not a realistic lattice, however, its amazing topology makes the exact calculations of various quantities possible. First attempts to investigate QMC over such trees was done in [16], such studies were related to the investigation of thermodynamic limit of valence-bond-solid models on a Cayley tree [25] (see also [14]).

The phase transition phenomena is crucial for quantum models over multi dimensional lattices [20],[28],[52, 23]. In [17] it was considered quantum phase transition for the two-dimensional Ising model using  $C^*$ -algebra approach. In [25] the VBS-model was considered on the Cayley tree. It was established the existence of the phase transition, for the model in terms of finitely correlated states, which describes ground states of the model. Note that more general structure of finitely correlated states was studied in [26]. We stress that finitely correlated states can be considered as quantum Markov chains. In [30, 39, 40, 42, 43] noncommutative extensions of classical Markov fields, associated with Ising and Potts models on a Cayley tree, were investigated. In the classical case, Markov fields on trees are also considered in [49, 50],[53]-[56].

There are several methods to investigate phase transition from mathematical point of view. Roughly speaking, for a given Hamiltonian on a quasi local algebra, to establish the existence of a phase transition it is necessary to find at least two different KMS-states associated with a

model (see for details [23]). In [25] it was proposed to study the phase transition in the class of finitely correlated states. In the present paper, for a given Hamiltonian we provide a more general construction (than [10, 11]) of QMC associated with the Hamiltonian. Namely, in this construction, the Hamiltonian exhibits nearest-neighbor and next-nearest-neighbor interactions (in the previous papers [11] the Hamiltonian contained only nearest-neighbor interactions), and the corresponding QMC is defined as a weak limit of finite volume states (which depend on the Hamiltonian) with boundary conditions, i.e. QMC depends on the boundary conditions. We remark that in this construction, one can observe some similarities with Gibbs measures. We stress that all previous considered examples (in the literature) of QMC are related to Hamiltonians with nearest-neighbor interactions. A main aim of the present paper is to prove the existence of the phase transition in the class of QMC when the Hamiltonian contains both kinds (nearest-neighbor and next-nearest-neighbor) of interactions at the same time. Note that this kind of models do not have one-dimensional analogous, i.e. the considered model persists only on trees. In classical setting, this kind of model was called the Ising model with competing interactions, and has been rigorously investigated in many papers (see for example [31, 42, 43, 46, 50, 51]). Our main result is the following theorem

**Theorem 1.1.** *For the Ising model with competing interactions (23), (25),  $J > 0$ ,  $\beta > 0$  on the Cayley tree of order two, the following statements hold:*

- (i) *if  $\Delta(\theta) \leq 0$ , then there is a unique QMC;*
- (ii) *if  $\Delta(\theta) > 0$ , then there occurs a phase transition.*

Here  $\Delta(\theta) = \theta^J(\theta^2 - 3) - 2\theta$ ,  $\theta = e^{2\beta}$ .

By the phase transition we mean the existence of two distinct QMC for the given family of interaction operators  $\{K_{<x,y>}\}$ ,  $\{L_{>x,y<}\}$  (see (23), (25)). Moreover, these states should be not quasi-equivalent and their supports do not overlap. Note that in our earlier papers (see [12, 13]) we have proved only non quasi-equivalence of the states. In this paper, we additionally prove that the corresponding states do not have overlapping supports. Hence, the main result of the present paper recover a main result of [13] as a particular case ( $J = 0$ ). To prove the main result of the paper, we first establish that the model exhibits three translation-invariant QMC  $\varphi_\alpha$ ,  $\varphi_1$  and  $\varphi_2$ , and we study several properties of the states  $\varphi_1$  and  $\varphi_2$ .

We notice that the state  $\varphi_\alpha$  corresponds to the disordered phase of the model. In [21, 35] it was established that the disordered phase of the Ising model on the Cayley tree of order  $k \geq 2$  is extremal if and only if  $\theta < 1/\sqrt{k}$ . But for the Ising model with competing interactions this kind of result still is unknown. In the present paper, we find sheds some light into this question. Namely, we will prove the following result.

**Theorem 1.2.** *The states  $\varphi_\alpha$  and  $\varphi_1$  are not quasi-equivalent.*

This result shows how the states relate to each other, which is even a new phenomena in the classical setting.

Let us outline the organization of the paper. After preliminary information (see Section 2), in Section 3 we provide a general construction of quantum Markov chains on Cayley tree. This construction is more general than considered in [12]. Moreover, in this section we give the definition of the phase transition. Using the provided construction, in Section 4 we consider the Ising model with competing interactions on the Cayley tree of order two. Section 5 is devoted to the existence of the three translation-invariant QMC  $\varphi_\alpha$ ,  $\varphi_1$  and  $\varphi_2$  corresponding to the model. Section 6 contains the proof of Theorem 1.1, namely, we first prove that states  $\varphi_1$  and  $\varphi_2$  do not have overlapping supports, and they are not quasi-equivalent. We stress that the states  $\varphi_1$  and  $\varphi_2$  are not product states, and therefore, their non quasi equivalence is

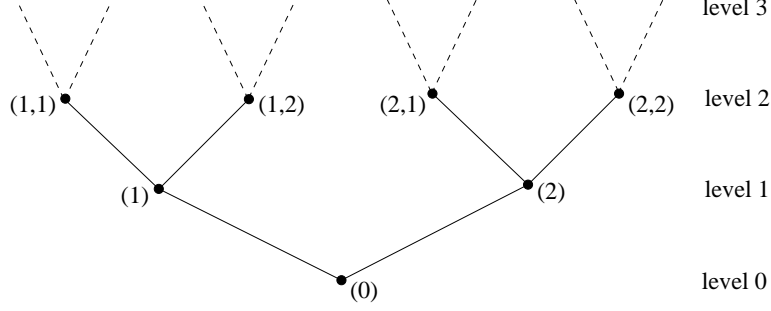


Figure 1: The first levels of  $\Gamma_+^2$

independent of interest from operator algebras point of view. In fact, in the case of product states, many papers were devoted to the study of quasi-equivalence [18, 37, 48]. In the final Section 7, we prove Theorem 1.2.

## 2 Preliminaries

Let  $\Gamma_+^k = (L, E)$  be a semi-infinite Cayley tree of order  $k \geq 1$  with the root  $x^0$  (i.e. each vertex of  $\Gamma_+^k$  has exactly  $k + 1$  edges, except for the root  $x^0$ , which has  $k$  edges). Here  $L$  is the set of vertices and  $E$  is the set of edges. The vertices  $x$  and  $y$  are called *nearest neighbors* and they are denoted by  $l = \langle x, y \rangle$  if there exists an edge connecting them. A collection of the pairs  $\langle x, x_1 \rangle, \dots, \langle x_{d-1}, y \rangle$  is called a *path* from the point  $x$  to the point  $y$ . The distance  $d(x, y)$ ,  $x, y \in V$ , on the Cayley tree, is the length of the shortest path from  $x$  to  $y$ .

Recall a coordinate structure in  $\Gamma_+^k$ : every vertex  $x$  (except for  $x^0$ ) of  $\Gamma_+^k$  has coordinates  $(i_1, \dots, i_n)$ , here  $i_m \in \{1, \dots, k\}$ ,  $1 \leq m \leq n$  and for the vertex  $x^0$  we put  $(0)$ . Namely, the symbol  $(0)$  constitutes level 0, and the sites  $(i_1, \dots, i_n)$  form level  $n$  (i.e.  $d(x^0, x) = n$ ) of the lattice (see Fig. 1).

Let us set

$$W_n = \{x \in L : d(x, x_0) = n\}, \quad \Lambda_n = \bigcup_{k=0}^n W_k, \quad \Lambda_{[n,m]} = \bigcup_{k=n}^m W_k, \quad (n < m)$$

$$E_n = \{ \langle x, y \rangle \in E : x, y \in \Lambda_n \}, \quad \Lambda_n^c = \bigcup_{k=n}^{\infty} W_k$$

For  $x \in \Gamma_+^k$ ,  $x = (i_1, \dots, i_n)$  denote

$$S(x) = \{(x, i) : 1 \leq i \leq k\}.$$

Here  $(x, i)$  means that  $(i_1, \dots, i_n, i)$ . This set is called a set of *direct successors* of  $x$ .

Two vertices  $x, y \in V$  is called *one level next-nearest-neighbor vertices* if there is a vertex  $z \in V$  such that  $x, y \in S(z)$ , and they are denoted by  $\succ x, y <$ . In this case the vertices  $x, z, y$  was called *ternary* and denoted by  $\langle x, z, y \rangle$ .

Let us define on  $\Gamma_+^k$  a binary operation  $\circ : \Gamma_+^k \times \Gamma_+^k \rightarrow \Gamma_+^k$  as follows: for any two elements  $x = (i_1, \dots, i_n)$  and  $y = (j_1, \dots, j_m)$  put

$$x \circ y = (i_1, \dots, i_n) \circ (j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m) \quad (1)$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \dots, i_n) \circ (0) = (i_1, \dots, i_n). \quad (2)$$

By means of the defined operation  $\Gamma_+^k$  becomes a noncommutative semigroup with a unit. Using this semigroup structure one defines translations  $\tau_g : \Gamma_+^k \rightarrow \Gamma_+^k$ ,  $g \in \Gamma_+^k$  by

$$\tau_g(x) = g \circ x. \quad (3)$$

It is clear that  $\tau_{(0)} = id$ .

The algebra of observables  $\mathcal{B}_x$  for any single site  $x \in L$  will be taken as the algebra  $M_d$  of the complex  $d \times d$  matrices. The algebra of observables localized in the finite volume  $\Lambda \subset L$  is then given by  $\mathcal{B}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}_x$ . As usual if  $\Lambda^1 \subset \Lambda^2 \subset L$ , then  $\mathcal{B}_{\Lambda^1}$  is identified as a subalgebra of  $\mathcal{B}_{\Lambda^2}$  by tensoring with unit matrices on the sites  $x \in \Lambda^2 \setminus \Lambda^1$ . Note that, in the sequel, by  $\mathcal{B}_{\Lambda,+}$  we denote the set of all positive elements of  $\mathcal{B}_\Lambda$  (note that an element is positive if its spectrum is located in  $\mathbb{R}_+$ ). The full algebra  $\mathcal{B}_L$  of the tree is obtained in the usual manner by an inductive limit

$$\mathcal{B}_L = \overline{\bigcup_{\Lambda_n} \mathcal{B}_{\Lambda_n}}.$$

In what follows, by  $\mathcal{S}(\mathcal{B}_\Lambda)$  we will denote the set of all states defined on the algebra  $\mathcal{B}_\Lambda$ .

Consider a triplet  $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$  of unital  $C^*$ -algebras. Recall [2] that a *quasi-conditional expectation* with respect to the given triplet is a completely positive (CP) linear map  $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\mathcal{E}(ca) = c\mathcal{E}(a)$ , for all  $a \in \mathcal{A}$ ,  $c \in \mathcal{C}$ .

**Definition 2.1** ([14]). *A state  $\varphi$  on  $\mathcal{B}_L$  is called a forward quantum Markov chain (QMC), associated to  $\{\Lambda_n\}$ , if for each  $\Lambda_n$ , there exist a quasi-conditional expectation  $\mathcal{E}_{\Lambda_n^c}$  with respect to the triplet*

$$\mathcal{B}_{\Lambda_{n+1}^c} \subseteq \mathcal{B}_{\Lambda_n^c} \subseteq \mathcal{B}_{\Lambda_{n-1}^c} \quad (4)$$

and a state  $\hat{\varphi}_{\Lambda_n^c} \in \mathcal{S}(\mathcal{B}_{\Lambda_n^c})$  such that for any  $n \in \mathbb{N}$  one has

$$\hat{\varphi}_{\Lambda_n^c} |_{\mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}} = \hat{\varphi}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_{n+1}^c} |_{\mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n}} \quad (5)$$

and

$$\varphi = \lim_{n \rightarrow \infty} \hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_{n-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c} \quad (6)$$

in the weak- $*$  topology.

Note that (5) is an analogue of the DRL equation from classical statistical mechanics [24, 32], and QMC is thus the counterpart of the infinite-volume Gibbs measure.

### 3 Construction of Quantum Markov Chains on Cayley tree

In this section we are going to provide a construction of a forward quantum Markov chain which contain competing interactions. Note that in our construction generalizes our previous works [11, 12].

Let us rewrite the elements of  $W_n$  in the following order, i.e.

$$\overrightarrow{W_n} := (x_{W_n}^{(1)}, x_{W_n}^{(2)}, \dots, x_{W_n}^{(|W_n|)}), \quad \overleftarrow{W_n} := (x_{W_n}^{(|W_n|)}, x_{W_n}^{(|W_n|-1)}, \dots, x_{W_n}^{(1)}).$$

Note that  $|W_n| = k^n$ . Vertices  $x_{W_n}^{(1)}, x_{W_n}^{(2)}, \dots, x_{W_n}^{(|W_n|)}$  of  $W_n$  can be represented in terms of the coordinate system as follows

$$\begin{aligned} x_{W_n}^{(1)} &= (1, 1, \dots, 1, 1), & x_{W_n}^{(2)} &= (1, 1, \dots, 1, 2), & \dots & & x_{W_n}^{(k)} &= (1, 1, \dots, 1, k), \\ x_{W_n}^{(k+1)} &= (1, 1, \dots, 2, 1), & x_{W_n}^{(2k)} &= (1, 1, \dots, 2, 2), & \dots & & x_{W_n}^{(2k)} &= (1, 1, \dots, 2, k), \end{aligned} \quad (7)$$

$\vdots$

$$x_{W_n}^{(|W_n|-k+1)} = (k, k, \dots, k, 1), \quad x_{W_n}^{(|W_n|-k+2)} = (k, k, \dots, k, 2), \quad \dots \quad x_{W_n}^{|W_n|} = (k, k, \dots, k, k).$$

Analogously, for a given vertex  $x$ , we shall use the following notation for the set of direct successors of  $x$ :

$$\overrightarrow{S(x)} := ((x, 1), (x, 2), \dots, (x, k)), \quad \overleftarrow{S(x)} := ((x, k), (x, k-1), \dots, (x, 1)).$$

In what follows, by  $\circ \prod$  we denote an ordered product, i.e.

$$\circ \prod_{k=1}^n a_k = a_1 a_2 \cdots a_n,$$

where elements  $\{a_k\} \subset \mathcal{B}_L$  are multiplied in the indicated order. This means that we are not allowed to change this order.

Notice that each vertex  $x \in L$  has interacting vertices  $\{x, (x, 1), \dots, (x, k)\}$ . Assume that each edges  $< x, (x, i) > (i = 1, \dots, k)$  operators  $K_{<x, (x, i)>} \in \mathcal{B}_x \otimes \mathcal{B}_{(x, i)}$  is assigned, respectively. Moreover, for each competing vertices  $> (x, i), (x, i+1) <$  and  $< x, (x, i), (x, i+1) > (i = 1, \dots, k)$  the following operators are assigned:

$$L_{>(x, i), (x, i+1)<} \in \mathcal{B}_{(x, i)} \otimes \mathcal{B}_{(x, i+1)}, \quad M_{(x, (x, i), (x, i+1))} \in \mathcal{B}_x \otimes \mathcal{B}_{(x, i)} \otimes \mathcal{B}_{(x, i+1)}.$$

We would like to define a state on  $\mathcal{B}_{\Lambda_n}$  with boundary conditions  $\omega_0 \in \mathcal{B}_{(0),+}$  and  $\{h^x \in \mathcal{B}_{x,+} : x \in L\}$ . Note that the boundary conditions have similar interpretations like classical ones (i.e. having boundary spins parallel either all up or all down) but in more general setting. For example, if we consider an Ising model on a Cayley tree, then the boundary conditions for this model are defined by functions  $\{h_x\}_{x \in V}$ , which in our setting correspond to  $\{\exp(h_x)\}_{x \in V}$ . For this reason, we are considering positive elements  $h^x \in \mathcal{B}_{x,+}$  as the boundary condition. For more information about the boundary conditions related to classical models we refer [29, 50].

For each  $n \in \mathbf{N}$  denote

$$A_{x, (x, 1), \dots, (x, k)} = \left( \circ \prod_{i=1}^k K_{x, (x, i)} \right) \left( \circ \prod_{i=1}^k L_{>(x, i), (x, i+1)<} \right) \left( \circ \prod_{i=1}^k M_{(x, (x, i), (x, i+1))} \right), \quad (8)$$

$$K_{[m, m+1]} := \prod_{x \in \vec{W}_m} A_{x, (x, 1), \dots, (x, k)}, \quad 1 \leq m \leq n, \quad (9)$$

$$\mathbf{h}_n^{1/2} := \prod_{x \in \vec{W}_n} (h^x)^{1/2}, \quad \mathbf{h}_n = \mathbf{h}_n^{1/2} (\mathbf{h}_n^{1/2})^* \quad (10)$$

$$\mathbf{K}_n := \omega_0^{1/2} \prod_{m=1}^{n-1} K_{[m, m+1]} \mathbf{h}_n^{1/2} \quad (11)$$

$$\mathcal{W}_n := \mathbf{K}_n \mathbf{K}_n^* \quad (12)$$

One can see that  $\mathcal{W}_n$  is positive.

In what follows, by  $\text{Tr}_\Lambda : \mathcal{B}_L \rightarrow \mathcal{B}_\Lambda$  we mean normalized partial trace (i.e.  $\text{Tr}_\Lambda(\mathbf{1}_L) = \mathbf{1}_\Lambda$ , here  $\mathbf{1}_\Lambda = \bigotimes_{y \in \Lambda} \mathbf{1}$ ), for any  $\Lambda \subseteq_{\text{fin}} L$ . For the sake of shortness we put  $\text{Tr}_n := \text{Tr}_{\Lambda_n}$ .

Let us define a positive functional  $\varphi_{w_0, \mathbf{h}}^{(n)}$  on  $\mathcal{B}_{\Lambda_n}$  by

$$\varphi_{w_0, \mathbf{h}}^{(n)}(a) = \text{Tr}(\mathcal{W}_{n+1}(a \otimes \mathbf{1}_{W_{n+1}})), \quad (13)$$

for every  $a \in \mathcal{B}_{\Lambda_n}$ . Note that here,  $\text{Tr}$  is a normalized trace on  $\mathcal{B}_L$  (i.e.  $\text{Tr}(\mathbf{1}_L) = 1$ ).

To get an infinite-volume state  $\varphi$  on  $\mathcal{B}_L$  such that  $\varphi|_{\mathcal{B}_{\Lambda_n}} = \varphi_{w_0, \mathbf{h}}^{(n)}$ , we need to impose some constraints to the boundary conditions  $\{w_0, \mathbf{h}\}$  so that the functionals  $\{\varphi_{w_0, \mathbf{h}}^{(n)}\}$  satisfy the compatibility condition, i.e.

$$\varphi_{w_0, \mathbf{h}}^{(n+1)}|_{\mathcal{B}_{\Lambda_n}} = \varphi_{w_0, \mathbf{h}}^{(n)}. \quad (14)$$

In the following we need an auxiliary fact.

**Lemma 3.1.** *Let  $\Lambda \subseteq \Lambda' \subseteq_{fin} L$ , then for any  $A \in \mathcal{B}_\Lambda, B \in \mathcal{B}_{\Lambda'}$  one has  $\text{Tr}(AB) = \text{Tr}[A \text{Tr}_{\mathcal{B}_\Lambda}(B)]$ .*

**Theorem 3.2.** *Assume that for every  $x \in L$  and triple  $\{x, (x, i), (x, i+1)\}$  ( $i = 1, \dots, k-1$ ) the operators  $K_{<x, (x, i)>}, L_{>(x, i), (x, i+1)<}, M_{(x, (x, i), (x, i+1))}$  are given as above. Let the boundary conditions  $w_0 \in \mathcal{B}_{(0), +}$  and  $\mathbf{h} = \{h_x \in \mathcal{B}_{x, +}\}_{x \in L}$  satisfy the following conditions:*

$$\text{Tr}(\omega_0 h^{(0)}) = 1, \quad (15)$$

$$\text{Tr}_x(A_{x, (x, 1), \dots, (x, k)} \circ \prod_{i=1}^k h^{(x, i)} A_{x, (x, 1), \dots, (x, k)}^*) = h^x, \quad \text{for every } x \in L, \quad (16)$$

where as before  $A_{x, (x, 1), \dots, (x, k)}$  is given by (8). Then the functionals  $\{\varphi_{w_0, \mathbf{h}}^{(n)}\}$  satisfy the compatibility condition (14). Moreover, there is a unique forward quantum Markov chain  $\varphi_{w_0, \mathbf{h}}$  on  $\mathcal{B}_L$  such that  $\varphi_{w_0, \mathbf{h}} = w - \lim_{n \rightarrow \infty} \varphi_{w_0, \mathbf{h}}^{(n)}$ .

*Proof.* We first show that a sequence  $\{\mathcal{W}_n\}$  is *projective* with respect to  $\text{Tr}_n$ , i.e.

$$\text{Tr}_{n-1}(\mathcal{W}_n) = \mathcal{W}_{n-1}, \quad \forall n \in \mathbb{N}. \quad (17)$$

It is known [8] that the projectivity implies the compatibility condition.

Now let us check the equality (17). From (8)-(12) one has

$$\mathcal{W}_n = w_0^{1/2} \left( \prod_{m=1}^{n-1} K_{[m-1, m]} \right) K_{[n-1, n]} \mathbf{h}_n K_{[n-1, n]}^* \left( \prod_{m=1}^{n-1} K_{[m-1, m]} \right)^* w_0^{1/2}.$$

One can see that for different  $x$  and  $x'$  taken from  $W_{n-1}$  the algebras  $\mathcal{B}_{x \cup S(x)}$  and  $\mathcal{B}_{x' \cup S(x')}$  are commuting, therefore from (10) one finds

$$K_{[n-1, n]} \mathbf{h}_n K_{[n-1, n]}^* = \prod_{x \in \vec{W}_{n-1}} (A_{x, (x, 1), \dots, (x, k)} \circ \prod_{i=1}^k h^{(x, i)} A_{x, (x, 1), \dots, (x, k)}^*)$$

Hence, from the last equality with (16) we get

$$\begin{aligned} \text{Tr}_{n-1}(\mathcal{W}_n) &= w_0^{1/2} \left( \prod_{m=1}^{n-1} K_{[m-1, m]} \right) \\ &\quad \times \prod_{x \in \vec{W}_{n-1}} \text{Tr}_x \left( A_{x, (x, 1), \dots, (x, k)} \circ \prod_{i=1}^k h^{(x, i)} A_{x, (x, 1), \dots, (x, k)}^* \right) \\ &\quad \times \left( \prod_{m=1}^{n-1} K_{[m-1, m]} \right)^* w_0^{1/2} \\ &= w_0^{1/2} \left( \prod_{m=1}^{n-1} K_{[m-1, m]} \right) \prod_{x \in \vec{W}_{n-1}} h^x \left( \prod_{m=1}^{n-1} K_{[m-1, m]} \right)^* w_0^{1/2} \\ &= \mathcal{W}_{n-1}. \end{aligned}$$

From the above argument and (15), one can show that  $\mathcal{W}_n$  is density operator, i.e.  $\text{Tr}(\mathcal{W}_n) = 1$ .

Let us show that the defined state  $\varphi_{w_0, \mathbf{h}}$  is a forward QMC. Indeed, define quasi-conditional expectations  $\mathcal{E}_{\Lambda_n^c}$  as follows:

$$\hat{\mathcal{E}}_{\Lambda_1^c}(x_{[0]}) = \text{Tr}_{[1]}(K_{[0,1]}w_0^{1/2}x_{[0]}w_0^{1/2}K_{[0,1]}^*), \quad x_{[0]} \in \mathcal{B}_{\Lambda_0^c} \quad (18)$$

$$\mathcal{E}_{\Lambda_k^c}(x_{[k-1]}) = \text{Tr}_{[n]}(K_{[k-1,k]}x_{[k-1]}K_{[k-1,k]}^*), \quad x_{[k-1]} \in \mathcal{B}_{\Lambda_{k-1}^c}, \quad k = 1, 2, \dots, n+1, \quad (19)$$

here  $\text{Tr}_{[n]} = \text{Tr}_{\Lambda_n^c}$ . Then for any monomial  $a_{\Lambda_1} \otimes a_{W_2} \otimes \dots \otimes a_{W_n} \otimes \mathbf{1}_{W_{n+1}}$ , where  $a_{\Lambda_1} \in \mathcal{B}_{\Lambda_1}$ ,  $a_{W_k} \in \mathcal{B}_{W_k}$ , ( $k = 2, \dots, n$ ), we have

$$\begin{aligned} \varphi_{w_0, \mathbf{h}}^{(n)}(a_{\Lambda_1} \otimes a_{W_2} \otimes \dots \otimes a_{W_n}) &= \text{Tr} \left( \mathbf{h}_{n+1} K_{[n,n+1]}^* \dots K_{[0,1]}^* w_0^{1/2} (a_{\Lambda_1} \otimes a_{W_2} \otimes \dots \otimes a_{W_n}) \right. \\ &\quad \left. w_0^{1/2} K_{[0,1]} \dots K_{[n,n+1]} \right) \\ &= \text{Tr}_{[1]} \left( \mathbf{h}_{n+1} K_{[n,n+1]}^* \dots K_{[1,2]}^* \hat{\mathcal{E}}_{\Lambda_1^c}(a_{\Lambda_1}) a_{W_2} K_{[1,2]} \right. \\ &\quad \left. \dots a_{W_n} K_{[n,n+1]} \right) \\ &= \text{Tr}_{[n+1]} (\mathbf{h}_{n+1} \mathcal{E}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \dots \\ &\quad \mathcal{E}_{\Lambda_2^c} \circ \hat{\mathcal{E}}_{\Lambda_1^c}(a_{\Lambda_1} \otimes a_{W_2} \otimes \dots \otimes a_{W_n})). \end{aligned} \quad (20)$$

Hence, for any  $a \in \Lambda \subset \Lambda_{n+1}$  from (13) with (9), (10), (18)-(20) one can see that

$$\varphi_{w_0, \mathbf{h}}^{(n)}(a) = \text{Tr}_{[n+1]} (\mathbf{h}_{n+1} \mathcal{E}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \dots \mathcal{E}_{\Lambda_2^c} \circ \hat{\mathcal{E}}_{\Lambda_1^c}(a)). \quad (21)$$

The projectivity of  $\mathcal{W}_n$  yields the equality (5) for  $\varphi_{w_0, \mathbf{h}}^{(n)}$ , therefore, from (21) we conclude that  $\varphi_{w_0, \mathbf{h}}$  is a forward QMC.  $\square$

**Corollary 3.3.** *If (15), (16) are satisfied then one has  $\varphi_{w_0, \mathbf{h}}^{(n)}(a) = \text{Tr}(\mathcal{W}_n(a))$  for any  $a \in \mathcal{B}_{\Lambda_n}$ .*

**Remark 3.4.** *If one takes  $L_{>(x,i),(x,i+1)<} = \mathbf{1}$ ,  $M_{(x,(x,i),(x,i+1))} = \mathbf{1}$  for all  $x \in L$ , then we get a QMC constructed in [11]. Therefore, the provided construction extensions ones given in [11, 12].*

Our goal in this paper is to establish the existence of phase transition for the given family of operators. Heuristically, the phase transition means the existence of two distinct QMC. Let us provide a more exact definition.

**Definition 3.5.** *We say that there exists a phase transition for a family of operators  $\{K_{<x,(x,i)>}\}$ ,  $\{L_{>(x,i),(x,i+1)<}\}$ ,  $\{M_{(x,(x,i),(x,i+1))}\}$  if the following conditions are satisfied:*

- (a) **EXISTENCE:** *The equations (15), (16) have at least two  $(u_0, \{h^x\}_{x \in L})$  and  $(v_0, \{s^x\}_{x \in L})$  solutions;*
- (c) **NOT OVERLAPPING SUPPORTS:** *there is a projector  $P \in B_L$  such that  $\varphi_{u_0, \mathbf{h}}(P) < \varepsilon$  and  $\varphi_{v_0, \mathbf{s}}(P) > 1 - \varepsilon$ , for some  $\varepsilon > 0$ .*
- (b) **NON QUASI-EQUIVALENCE:** *the corresponding quantum Markov chains  $\varphi_{u_0, \mathbf{h}}$  and  $\varphi_{v_0, \mathbf{s}}$  are not quasi equivalent<sup>1</sup>.*

---

<sup>1</sup>Recall that a representation  $\pi_1$  of a  $C^*$ -algebra  $\mathfrak{A}$  is *normal* w.r.t. another representation  $\pi_2$ , if there is a normal  $*$ -epimorphism  $\rho: \pi_2(\mathfrak{A})'' \rightarrow \pi_1(\mathfrak{A})''$  such that  $\rho \circ \pi_2 = \pi_1$ . Two representations  $\pi_1$  and  $\pi_2$  are called *quasi-equivalent* if  $\pi_1$  is normal w.r.t.  $\pi_2$ , and conversely,  $\pi_2$  is normal w.r.t.  $\pi_1$ . This means that there is an isomorphism  $\alpha: \pi_1(\mathfrak{A})'' \rightarrow \pi_2(\mathfrak{A})''$  such that  $\alpha \circ \pi_1 = \pi_2$ . Two states  $\varphi$  and  $\psi$  of  $\mathfrak{A}$  are said be *quasi-equivalent* if the GNS representations  $\pi_\varphi$  and  $\pi_\psi$  are quasi-equivalent [22].



Otherwise, we say there is no phase transition.

**Remark 3.6.** We stress that in [12] we have first introduced a notion of the phase transition quantum Markov chains. That definition is contained only (a) and (b) conditions. After some discussions it was observed that these two conditions are not sufficient for the existence of the phase transition. Since, the non quasi equivalence of the states does not imply that the states are different. Namely, it might happen that one of the states could be absolutely continuous to another one. In general, to have a phase transition the states should not be absolutely continuous to each other. Therefore, the third condition (c) is imposed in the present definition.

## 4 QMC associated with Ising model with competing interactions

In this section, we define the model and formulate the main results of the paper. In what follows we consider a semi-infinite Cayley tree  $\Gamma_+^2 = (L, E)$  of order two. Our starting  $C^*$ -algebra is the same  $\mathcal{B}_L$  but with  $\mathcal{B}_x = M_2(\mathbb{C})$  for all  $x \in L$ . Denote

$$\mathbf{1}^{(u)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^{(u)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (22)$$

For every vertices  $(x, (x, 1), (x, 2))$  we put

$$K_{<x, (x, i)>} = \exp\{\beta H_{<x, (x, i)>}\}, \quad i = 1, 2, \quad \beta > 0, \quad (23)$$

$$L_{>(x, 1), (x, 2)<} = \exp\{J\beta H_{>(x, 1), (x, 2)<}\}, \quad J > 0, \quad (24)$$

where

$$H_{<x, (x, i)>} = \frac{1}{2}(\mathbf{1}^{(x)} \mathbf{1}^{(x, i)} + \sigma^{(x)} \sigma^{(x, i)}), \quad (25)$$

$$H_{>(x, 1), (x, 2)<} = \frac{1}{2}(\mathbf{1}^{(x, 1)} \mathbf{1}^{(x, 2)} + \sigma^{(x, 1)} \sigma^{(x, 2)}). \quad (26)$$

Furthermore, we assume that  $M_{(x, (x, i), (x, i+1))} = \mathbf{1}$  ( $i = 1, 2, \dots, k$ ) for all  $x \in L$ .

The defined model is called the *Ising model with competing interactions* per vertices  $(x, (x, 1), (x, 2))$ .

**Remark 4.1.** Note that if we take  $J = 0$ , then one gets the Ising model on Cayley tree which has been studied in [13]. In [41] we have studied the classical Ising model with competing interactions in comparison with quantum analogous.

One can calculate that

$$H_{<u, v>}^m = H_{<u, v>} = \frac{1}{2}(\mathbf{1}^{(u)} \mathbf{1}^{(v)} + \sigma^{(u)} \sigma^{(v)}), \quad (27)$$

$$H_{>x, y<}^m = H_{>x, y<} = \frac{1}{2}(\mathbf{1}^{(x)} \mathbf{1}^{(y)} + \sigma^{(x)} \sigma^{(y)}). \quad (28)$$

Therefore, one finds

$$K_{<u, v>} = K_0 \mathbf{1}^{(u)} \mathbf{1}^{(v)} + K_3 \sigma^{(u)} \sigma^{(v)}, \quad (29)$$

$$L_{>u, v<} = R_0 \mathbf{1}^{(u)} \mathbf{1}^{(v)} + R_3 \sigma^{(u)} \sigma^{(v)}, \quad (30)$$

where

$$K_0 = \frac{\exp \beta + 1}{2}, \quad K_3 = \frac{\exp \beta - 1}{2},$$

$$R_0 = \frac{\exp(J\beta) + 1}{2}, \quad R_3 = \frac{\exp(J\beta) - 1}{2}.$$

Hence, from (8) for each  $x \in L$  we obtain

$$\begin{aligned} A_{(x,(x,1),(x,2))} &= \gamma \mathbf{1}^{(x)} \otimes \mathbf{1}^{(x,1)} \otimes \mathbf{1}^{(x,2)} + \delta \sigma^{(x)} \otimes \sigma^{(x,1)} \otimes \mathbf{1}^{(x,2)} \\ &\quad + \delta \sigma^{(x)} \otimes \mathbf{1}^{(x,1)} \otimes \sigma^{(x,2)} + \eta \mathbf{1}^{(x)} \otimes \sigma^{(x,1)} \otimes \sigma^{(x,2)}, \end{aligned} \quad (31)$$

where

$$\begin{cases} \gamma = K_0^2 R_0 + K_3^2 R_3 = \frac{1}{4}[\exp(J+2)\beta + \exp J\beta + 2\exp \beta], \\ \delta = K_0 K_3 (R_0 + R_3) = \frac{1}{4} \exp J\beta [\exp 2\beta - 1], \\ \eta = K_0^2 R_3 + K_3^2 R_0 = \frac{1}{4}[\exp(J+2)\beta + \exp J\beta - 2\exp \beta]. \end{cases} \quad (32)$$

Recall that a function  $\{h^x\}$  is called *translation-invariant* if one has  $h^x = h^{\tau_g(x)}$ , for all  $x, g \in \Gamma_+^2$ . Clearly, this is equivalent to  $h^x = h^y$  for all  $x, y \in L$ .

In what follows, we restrict ourselves to the description of translation-invariant solutions of (15),(16). Therefore, we assume that:  $h^x = h$  for all  $x \in L$ , where

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Then we have

$$\begin{aligned} &A_{(x,(x,1),(x,2))} \times [\mathbf{1}^{(x)} \otimes h^{(x,1)} \otimes h^{(x,2)}] \times A_{(x,(x,1),(x,2))} \\ &= [\gamma^2 \mathbf{1} \otimes h \otimes h + \delta^2 \mathbf{1} \otimes \sigma h \sigma \otimes h + \delta^2 \mathbf{1} \otimes h \otimes \sigma h \sigma + \eta^2 \mathbf{1} \otimes \sigma h \sigma \otimes \sigma h \sigma] \\ &\quad + [\gamma \eta \mathbf{1} \otimes h \sigma \otimes h \sigma + \gamma \eta \mathbf{1} \otimes \sigma h \otimes \sigma h + \delta^2 \mathbf{1} \otimes \sigma h \otimes h \sigma + \delta^2 \mathbf{1} \otimes h \sigma \otimes \sigma h] \\ &\quad + [\gamma \delta \sigma \otimes h \sigma \otimes h + \gamma \delta \sigma \otimes h \otimes h \sigma + \gamma \delta \sigma \otimes h \otimes \sigma h + \gamma \delta \sigma \otimes \sigma h \otimes h] \\ &\quad + [\delta \eta \sigma \otimes \sigma h \sigma \otimes h \sigma + \delta \eta \sigma \otimes h \sigma \otimes \sigma h \sigma + \delta \eta \sigma \otimes \sigma h \sigma \otimes \sigma h + \delta \eta \sigma \otimes \sigma h \otimes \sigma h \sigma]. \end{aligned}$$

Hence, from the last equality we can rewrite (16) as follows

$$\begin{aligned} h &= \text{Tr}_x] A_{(x,(x,1),(x,2))} [\mathbf{1}^{(x)} \otimes h \otimes h] A_{(x,(x,1),(x,2))}^* \\ &= \tau_1 \text{Tr}(h)^2 + \tau_2 \text{Tr}(\sigma h)^2 \mathbf{1}^{(x)} + \tau_3 \text{Tr}(h) \text{Tr}(\sigma h) \sigma^{(x)}. \end{aligned} \quad (33)$$

Here  $\theta = \exp 2\beta > 0$  and

$$\begin{cases} \tau_1 := \gamma^2 + 2\delta^2 + \eta^2 = \frac{1}{4}[\theta^J(\theta^2 + 1) + 2\theta], \\ \tau_2 := 2(\gamma\eta + \delta^2) = \frac{1}{4}[\theta^J(\theta^2 + 1) - 2\theta], \\ \tau_3 := 4\delta(\gamma + \eta) = \frac{1}{2}\theta^J(\theta^2 - 1), \end{cases}$$

Now taking into account

$$\mathrm{Tr}(h) = \frac{h_{11} + h_{22}}{2}, \quad \mathrm{Tr}(\sigma h) = \frac{h_{11} - h_{22}}{2}$$

the equation (33) is reduced to the following one

$$\begin{cases} \mathrm{Tr}(h) = \tau_1 \mathrm{Tr}(h)^2 + \tau_2 \mathrm{Tr}(\sigma h)^2, \\ \mathrm{Tr}(\sigma h) = \tau_3 \mathrm{Tr}(h) \mathrm{Tr}(\sigma h), \\ h_{21} = 0, h_{12} = 0. \end{cases} \quad (34)$$

The obtained equation implies that a solution  $h$  is diagonal, and  $\omega_0$  could be also chosen diagonal, through the equation. In what follows, we always assume that  $h_{21} = 0, h_{12} = 0$ . In the next sections we are going to examine (34).

## 5 Existence of QMC associated with the model.

In this section we are going to solve (34), which yields the existence of QMC associated with the model.

### 5.1 Case $h_{11} = h_{22}$ and associate QMC

Assume that  $h_{1,1} = h_{2,2}$ , then (34) is reduced to

$$h_{11} = h_{22} = \frac{1}{\tau_1}.$$

Then putting  $\alpha = \frac{1}{\tau_1}$  we get

$$h_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad (35)$$

**Proposition 5.1.** *The pair  $(\omega_0, \{h^x = h_\alpha | x \in L\})$  with  $\omega_0 = \frac{1}{\alpha} \mathbf{1}$ ,  $h^x = h_\alpha, \forall x \in L$ , is solution of (15),(16). Moreover the associated QMC can be written on the local algebra  $\mathcal{B}_{L,loc}$  by:*

$$\varphi_\alpha(a) = \alpha^{2^n-1} \mathrm{Tr} \left( a \prod_{i=0}^{n-1} K_{[i,i+1]} K_{[i,i+1]}^* \right), \quad \forall a \in B_{\Lambda_n}. \quad (36)$$

*Proof.* Let  $n \in \mathbb{N}, a \in B_{\Lambda_n}$  then using Lemma 3.1 one finds

$$\begin{aligned} \varphi_\alpha(a) &= \mathrm{Tr} \left( \omega_0 h_n \left( \prod_{m=0}^{n-1} K_{[m,m+1]} K_{[m,m+1]}^* \right) a \right) \\ &= \alpha^{|W_n|-1} \mathrm{Tr} \left( a \prod_{m=0}^{n-1} K_{[m,m+1]} K_{[m,m+1]}^* \right) \\ &= \alpha^{2^n-1} \mathrm{Tr} \left( a \prod_{i=0}^{n-1} K_{[i,i+1]} K_{[i,i+1]}^* \right). \end{aligned}$$

This completes the proof. □

## 5.2 Case $h_{11} \neq h_{22}$ and associate QMC

Assume that  $h_{11} \neq h_{22}$ , then (34) is reduced to

$$\begin{cases} h_{11} + h_{22} = \frac{1}{\tau_2}, \\ (h_{11} - h_{22})^2 = \frac{\tau_3 - \tau_1}{\tau_2 \cdot \tau_3^2}, \end{cases} \quad (37)$$

Let

$$\Delta(\theta) := 4(\tau_3 - \tau_1) = \theta^J(\theta^2 - 3) - 2\theta$$

One can see that the last system has a solution iff  $\tau_3 > \tau_1$ , i.e. whenever  $\Delta(\theta) > 0$ .

**Proposition 5.2.** *Assume that  $\tau_3 > \tau_1$ . Then the equation (34) has two solutions given by:*

$$h = \xi_0 \mathbf{1} + \xi_3 \sigma, \quad (38)$$

$$h' = \xi_0 \mathbf{1} - \xi_3 \sigma, \quad (39)$$

where

$$\xi_0 = \frac{1}{\tau_3} = \frac{2}{\theta^J(\theta^2 - 1)}, \quad \xi_3 = \frac{\sqrt{\tau_3 - \tau_1}}{\tau_3 \sqrt{\tau_2}} = \frac{2}{\theta^J(\theta^2 - 1)} \sqrt{\frac{\Delta(\theta)}{\theta^J(\theta^2 + 1) - 2\theta}} \quad (40)$$

*Proof.* Assume that  $\Delta(\theta) > 0$ . Then one can conclude that (37) is equivalent to the following system

$$\begin{cases} h_{1,1} + h_{2,2} = 2\xi_0, \\ h_{1,1} - h_{2,2} = \pm 2\xi_3 \end{cases}$$

It is easy to see that  $h_{1,1} = \xi_0 - \xi_3$ ,  $h_{2,2} = \xi_0 + \xi_3$ . Hence, we get (38),(39).  $\square$

From (15) we find that  $\omega_0 = \frac{1}{\xi_0} \mathbf{1} \in \mathcal{B}^+_{\text{s}}$ . Therefore, the pairs  $(\omega_0, \{h^{(x)} = h, x \in L\})$  and  $(\omega_0, \{h^{(x)} = h', x \in L\})$  define two solutions of (15),(16). Hence, they define two QMC  $\varphi_1$  and  $\varphi_2$ , respectively. Namely, for every  $a \in \mathcal{B}_{\Lambda_n}$  one has

$$\varphi_1(a) = \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} \mathbf{h}_n K_{[n-1,n]}^* \cdots K_{[0,1]}^* a) \quad (41)$$

$$\varphi_2(a) = \text{Tr}(\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} \mathbf{h}'_n K_{[n-1,n]}^* \cdots K_{[0,1]}^* a). \quad (42)$$

Hence, we have proved the following

**Theorem 5.3.** *Let  $\theta > 1$ . Then the following statements hold:*

- (i) *if  $\Delta(\theta) \leq 0$ , then there is a unique translation invariant QMC  $\varphi_\alpha$ ;*
- (ii) *if  $\Delta(\theta) > 0$ , then there are at least three translation invariant QMC  $\varphi_\alpha$ ,  $\varphi_1$  and  $\varphi_2$ .*

We point out that the critical line  $\Delta(\theta) = 0$  is first observed in [31] for the classical Ising model with competing interactions.

Now let us consider some particular values of  $\theta$ .

(i) Let  $J = 1$ , then one has  $\Delta(\theta) = \theta^3 - 5\theta$ . So,

- if  $1 < \theta \leq \sqrt{5}$ , there is a unique QMC;
- if  $\theta > \sqrt{5}$ , there exist three QMC.

(ii) Let  $J = 2$ , then  $\Delta(\theta) = \theta(\theta + 1)(\theta - \frac{1-\sqrt{5}}{2})(\theta - \frac{1+\sqrt{5}}{2})$ . Hence,

- if  $1 < \theta \leq \frac{1+\sqrt{5}}{2}$ , there is a unique QMC;
- if  $\theta > \frac{1+\sqrt{5}}{2}$ , there exist three QMC.

## 6 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To realize it, we first show not overlapping supports of the states  $\varphi_1$  and  $\varphi_2$ . Then we will show that they are not quasi-equivalent.

### 6.1 Not overlapping supports of $\varphi_1$ and $\varphi_2$

As usual we put

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now for each  $n \in \mathbb{N}$  denote

$$p_n := \left( \bigotimes_{x \in \Lambda_n} e_{11}^{(x)} \right) \otimes \mathbf{1}, \quad q_n := \left( \bigotimes_{x \in \Lambda_n} e_{22}^{(x)} \right) \otimes \mathbf{1}.$$

**Lemma 6.1.** *For every  $n \in \mathbb{N}$  one has*

$$(i) \quad \varphi_1(p_n) = \varphi_2(q_n) = \frac{1}{\xi_0} (\xi_0 + \xi_3)^{2^n} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{2^n - 1},$$

$$(ii) \quad \varphi_1(q_n) = \varphi_2(p_n) = \frac{1}{\xi_0} (\xi_0 - \xi_3)^{2^n} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{2^n - 1}.$$

*Proof.* (i). From (41) we find

$$\varphi_1(p_n) = \text{Tr} \left[ \omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} \cdot \text{Tr}_{n-1} \left( (K_{[n-1,n]} \mathbf{h}_n p_n K_{[n-1,n]}^*) \cdot K_{[n-2,n-1]}^* \cdots K_{[0,1]}^* \right) \right]$$

Now using the fact that:  $h e_{11} = (\xi_0 + \xi_3) e_{11}$  and (31) one gets

$$\begin{aligned} \text{Tr}_{n-1} K_{[n-1,n]} \mathbf{h}_n p_n K_{[n-1,n]}^* &= p_{n-2} \otimes \prod_{x \in \vec{W}_{n-1}} A_{(x,(x,1),(x,2))} e_{11}^{(x)} \otimes h e_{11}^{(x,1)} \otimes h e_{11}^{(x,2)} A_{(x,(x,1),(x,2))} \\ &= (\xi_0 + \xi_3)^{2|W_{n-1}|} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{|W_{n-1}|} p_{n-1}. \end{aligned}$$

Hence,

$$\begin{aligned} \varphi_1(p_n) &= (\xi_0 + \xi_3)^{|W_n|} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{|W_{n-1}|} \text{Tr} \left[ \omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} p_{n-1} K_{[n-2,n-1]}^* \cdots K_{[0,1]}^* \right] \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &= (\xi_0 + \xi_3)^{|W_n|} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{|W_{n-1}| + \dots + |W_0|} \cdot \text{Tr} [\omega_0 p_0] \\ &= (\xi_0 + \xi_3)^{|W_n|} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{|\Lambda_{n-1}|} \text{Tr} [\omega_0 p_0] \\ &= \frac{1}{\xi_0} (\xi_0 + \xi_3)^{2^n} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{2^n - 1} \left( \frac{1}{\xi_0} + \frac{1}{\xi_3} \right). \end{aligned}$$

Analogously, using the fact that:  $h' e_{22} = (\xi_0 + \xi_3) e_{22}$  we obtain

$$\begin{aligned}
\text{Tr}_{n-1}] K_{[n-1,n]} \mathbf{h}'_n q_n K_{[n-1,n]}^* &= q_{n-2} \otimes \prod_{x \in \mathcal{W}_{n-1}} A_{(x,(x,1),(x,2))} e_{2,2}^{(x)} \otimes h' e_{2,2}^{(x,1)} \otimes h' e_{22}^{(x,2)} A_{(x,(x,1),(x,2))} \\
&= (\xi_0 + \xi_3)^{2|\mathcal{W}_{n-1}|} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{|\mathcal{W}_{n-1}|} q_{n-1}.
\end{aligned}$$

which yields

$$\varphi_2(q_n) = \frac{1}{\xi_0} (\xi_0 + \xi_3)^{2^n} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{2^{n-1}} \left( \frac{1}{\xi_0} + \frac{1}{\xi_3} \right).$$

(ii) Now from  $h e_{22} = (\xi_0 - \xi_3) e_{22}$  and  $h' e_{11} = (\xi_0 - \xi_3) e_{11}$ . One can find.

$$\begin{aligned}
\text{Tr}_{n-1}] K_{[n-1,n]} \mathbf{h}_n q_n K_{[n-1,n]}^* &= (\xi_0 - \xi_3)^{2|\mathcal{W}_{n-1}|} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{|\mathcal{W}_{n-1}|} q_{n-1}. \\
\text{Tr}_{n-1}] K_{[n-1,n]} \mathbf{h}'_n p_n K_{[n-1,n]}^* &= (\xi_0 - \xi_3)^{2|\mathcal{W}_{n-1}|} \left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{|\mathcal{W}_{n-1}|} p_{n-1}.
\end{aligned}$$

The same argument as above implies (ii). This completes the proof.  $\square$

**Theorem 6.2.** *For  $n \in \mathbf{N}$  fixed, one has*

$$\varphi_1(p_n) \rightarrow 1, \quad \varphi_2(q_n) \rightarrow 1 \quad \text{as } \beta \rightarrow +\infty.$$

*Proof.* We know that  $\theta = \exp(2\beta) \rightarrow +\infty$  as  $\beta \rightarrow +\infty$ . Hence, one finds

$$\begin{aligned}
\frac{1}{\xi_0} &= \frac{\theta^J(\theta^2 - 1)}{2} \sim \frac{\theta^{J+2}}{2}, \quad \text{as } \theta \rightarrow +\infty \\
(\xi_0 + \xi_3)^{2^n} &= \left( \frac{2}{\theta^J(\theta^2 - 1)} \left( 1 + \sqrt{\frac{\theta^J(\theta^2 - 3) - 2\theta}{\theta^J(\theta^2 + 1) - 2\theta}} \right) \right)^{2^n} \sim \left( \frac{4}{\theta^{J+2}} \right)^{2^n}, \quad \text{as } \theta \rightarrow +\infty \\
\left( \frac{\tau_1 + \tau_2 + \tau_3}{4} \right)^{2^{n-1}} &= \left( \frac{\theta^{J+2}}{4} \right)^{2^{n-1}}.
\end{aligned}$$

Hence, we obtain

$$\varphi_1(p_n) = \varphi_2(q_n) \sim \frac{\theta^{J+2}}{4} \left( \frac{4}{\theta^{J+2}} \right)^{2^n} \left( \frac{\theta^{J+2}}{4} \right)^{2^{n-1}} = 1.$$

So,

$$\lim_{\theta \rightarrow \infty} \varphi_1(p_n) = \lim_{\theta \rightarrow \infty} \varphi_2(q_n) = 1.$$

This completes the proof.  $\square$

**Remark 6.3.** *We note that from  $p_n \leq 1 - q_n$  one gets*

$$\lim_{\theta \rightarrow \infty} \varphi_1(q_n) = \lim_{\theta \rightarrow \infty} \varphi_2(p_n) = 0.$$

*This implies that the states  $\varphi_1$  and  $\varphi_2$  have not overlapping supports.*

## 6.2 Non quasi equivalence of $\varphi_1$ and $\varphi_2$

In this subsection we are going to proof that the state  $\varphi_1$  and  $\varphi_2$  are not quasi equivalent.

First note that from the construction of the states  $\varphi_1$  and  $\varphi_2$  one can see that they are translation invariant, i.e.  $\varphi_i \tau_g = \varphi_i$  ( $i = 1, 2$ ) (see (3)) for all  $g \in \Gamma_+^2$ . Moreover, using the same argument as in [54] one can show that states  $\varphi_1$  and  $\varphi_2$  satisfy mixing property, i.e.

$$\lim_{|g| \rightarrow \infty} \varphi_i(\tau_g(x)y) = \varphi_i(x)\varphi_i(y), \quad i = 1, 2.$$

This means that they are factor states.

To establish the non-quasi equivalence, we are going to use the following result (see [22, Corollary 2.6.11]).

**Theorem 6.4.** *Let  $\varphi_1, \varphi_2$  be two factor states on a quasi-local algebra  $\mathfrak{A} = \cup_{\Lambda} \mathfrak{A}_{\Lambda}$ . The states  $\varphi_1, \varphi_2$  are quasi-equivalent if and only if for any given  $\varepsilon > 0$  there exists a finite volume  $\Lambda \subset L$  such that  $\|\varphi_1(a) - \varphi_2(a)\| < \varepsilon \|a\|$  for all  $a \in B_{\Lambda'}$  with  $\Lambda' \cap \Lambda = \emptyset$ .*

Let us define an element of  $\mathcal{B}_{\Lambda_n}$  as follows:

$$E_{\Lambda_n} := e_{11}^{x_{W_n}^{(1)}} \otimes \left( \bigotimes_{y \in \Lambda_n \setminus \{x_{W_n}^{(1)}\}} \mathbb{I}^y \right),$$

where  $x_{W_n}^{(1)}$  is defined in (7).

Now we are going to calculate  $\varphi_1(E_{\Lambda_n})$  and  $\varphi_2(E_{\Lambda_n})$ , respectively.

First consider the state  $\varphi_1$ , then we know that this state is defined by  $\omega_0 = \tau_3 \mathbb{I}$  and  $h^x = h = \xi_0 \mathbb{I} + \xi_3 \sigma$ . Define two elements of  $\mathcal{B}_{W_n}$  by

$$\begin{aligned} \hat{\mathbf{h}}_n &:= \mathbb{I}^{x_{W_n}^{(1)}} \otimes \bigotimes_{x \in W_n \setminus \{x_{W_n}^{(1)}\}} h^{(x)} \\ \check{\mathbf{h}}_n &:= \sigma^{x_{W_n}^{(1)}} \otimes \bigotimes_{x \in W_n \setminus \{x_{W_n}^{(1)}\}} h^{(x)} \end{aligned}$$

**Lemma 6.5.** *Let*

$$\begin{aligned} \hat{\psi}_n &:= \text{Tr}_{n-1} [\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} \hat{\mathbf{h}}_n K_{[n-1,n]}^* \cdots K_{[0,1]}^*] \\ \check{\psi}_n &:= \text{Tr}_{n-1} [\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} \check{\mathbf{h}}_n K_{[n-1,n]}^* \cdots K_{[0,1]}^*] \end{aligned}$$

*Then there are two pairs of reals  $(\hat{\rho}_1, \hat{\rho}_2)$  and  $(\check{\rho}_1, \check{\rho}_2)$  depending on  $\theta$  such that*

$$\begin{cases} \hat{\psi}_n = \hat{\rho}_1 + \hat{\rho}_2 \left( \frac{\tau_1}{\tau_3} - 1 \right)^n, \\ \check{\psi}_n = \check{\rho}_1 + \check{\rho}_2 \left( \frac{\tau_1}{\tau_3} - 1 \right)^n \end{cases}$$

*Proof.* One can see that

$$\begin{pmatrix} \hat{\psi}_n \\ \check{\psi}_n \end{pmatrix} = \begin{pmatrix} \text{Tr}_n [\omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} \text{Tr}_{n-1} [K_{[n-1,n]} \hat{\mathbf{h}}_n K_{[n-1,n]}^*] K_{[n-2,n-1]}^* \cdots K_{[0,1]}^*], \\ \text{Tr}_n [\omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} \text{Tr}_{n-1} [K_{[n-1,n]} \check{\mathbf{h}}_n K_{[n-1,n]}^*] K_{[n-2,n-1]}^* \cdots K_{[0,1]}^*] \end{pmatrix}.$$

After small calculations, we find

$$\begin{cases} \text{Tr}_x [A_{(x,(x,1),(x,2))}(\mathbf{1}^{(x)} \otimes \mathbf{1}^{(x,1)} \otimes h^{(x,2)})A_{(x,(x,1),(x,2))}] = \tau_1 \xi_0 \mathbf{1}^{(x)} + \frac{1}{2} \tau_3 \xi_3 \sigma^{(x)} \\ \text{Tr}_x [A_{(x,(x,1),(x,2))}(\mathbf{1}^{(x)} \otimes \sigma^{(x,1)} \otimes h_{(\xi_0, \xi_3)}^{(x,2)})A_{(x,(x,1),(x,2))}] = \tau_2 \xi_3 \mathbf{1}^{(x)} + \frac{1}{2} \sigma^{(x)} \end{cases}$$

Hence, one gets

$$\begin{cases} \text{Tr}_{n-1} [K_{[n-1,n]} \hat{\mathbf{h}}_n K_{[n-1,n]}^*] = \tau_1 \xi_0 \hat{h}_{n-1} + \frac{1}{2} \tau_3 \xi_3 \check{h}_{n-1}, \\ \text{Tr}_{n-1} [K_{[n-1,n]} \check{\mathbf{h}}_n K_{[n-1,n]}^*] = \tau_2 \xi_3 \hat{h}_{n-1} + \frac{1}{2} \check{h}_{n-1}. \end{cases}$$

Therefore,

$$\begin{aligned} \begin{pmatrix} \hat{\psi}_n \\ \check{\psi}_n \end{pmatrix} &= \begin{pmatrix} \tau_1 \xi_0 \hat{\psi}_{n-1} + \frac{1}{2} \tau_3 \xi_3 \check{\psi}_{n-1} \\ \tau_2 \xi_3 \hat{\psi}_{n-1} + \frac{1}{2} \check{\psi}_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \tau_1 \xi_0 & \frac{1}{2} \tau_3 \xi_3 \\ \tau_2 \xi_3 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{\psi}_{n-1} \\ \check{\psi}_{n-1} \end{pmatrix} \\ &\vdots \\ &= \begin{pmatrix} \tau_1 \xi_0 & \frac{1}{2} \tau_3 \xi_3 \\ \tau_2 \xi_3 & \frac{1}{2} \end{pmatrix}^n \begin{pmatrix} \hat{\psi}_0 \\ \check{\psi}_0 \end{pmatrix}, \end{aligned}$$

where

$$\begin{cases} \hat{\psi}_0 = \text{Tr}(\omega_0) = \frac{1}{\xi_0} \\ \check{\psi}_0 = \text{Tr}(\omega_0 \cdot \sigma) = 0 \end{cases}$$

The matrix

$$N := \begin{pmatrix} \tau_1 \xi_0 & \frac{1}{2} \tau_3 \xi_3 \\ \tau_2 \xi_3 & \frac{1}{2} \end{pmatrix}$$

can be written in diagonal form by:

$$N = P \begin{pmatrix} 1 & 0 \\ 0 & \frac{\tau_1}{\tau_3} - \frac{1}{2} \end{pmatrix} P^{-1}$$

where

$$P = \begin{pmatrix} \frac{\tau_3}{2\tau_2} & -\frac{\xi_3}{\xi_0} \\ \frac{\xi_3}{\xi_0} & 1 \end{pmatrix}, \quad \det(P) = \frac{3\tau_3 - 2\tau_1}{2\tau_2}$$

So,

$$\begin{aligned} \begin{pmatrix} \hat{\psi}_n \\ \check{\psi}_n \end{pmatrix} &= P \begin{pmatrix} 1 & 0 \\ 0 & (\frac{\tau_1}{\tau_3} - \frac{1}{2})^n \end{pmatrix} P^{-1} \begin{pmatrix} \frac{1}{\xi_0} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \hat{\rho}_1 + \hat{\rho}_2 (\frac{\tau_1}{\tau_3} - \frac{1}{2})^n \\ \check{\rho}_1 + \check{\rho}_2 (\frac{\tau_1}{\tau_3} - \frac{1}{2})^n \end{pmatrix}. \end{aligned}$$

where

$$\hat{\rho}_1 = \frac{2\tau_3^2}{3\tau_3 - 2\tau_1}, \quad \hat{\rho}_2 = \frac{2\tau_3(\tau_3 - \tau_1)}{3\tau_3 - 2\tau_1}, \quad (43)$$

$$\check{\rho}_1 = \frac{2\tau_2\tau_3^2\xi_3}{3\tau_3 - 2\tau_2}, \quad \check{\rho}_2 = -\frac{2\tau_2\tau_3^2\xi_3}{3\tau_3 - 2\tau_2}. \quad (44)$$

This completes the proof.  $\square$



**Proposition 6.6.** *For each  $n \in \mathbb{N}$  one has*

$$\begin{aligned}\varphi_1(E_{\Lambda_n}) &= \frac{1}{2} \left[ (\xi_0 + \xi_3)(\xi_0\tau_2 + \xi_3\tau_2)\hat{\rho}_1 + \frac{\tau_3}{2}(\xi_0 + \xi_3)^2\check{\rho}_1 \right] \\ &\quad + \frac{1}{2} \left[ (\xi_0 + \xi_3)(\xi_0\tau_1 + \xi_3\tau_2)\hat{\rho}_2 + \frac{1}{2}\tau_3(\xi_0 + \xi_3)^2\check{\rho}_2 \right] \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^{n-1}.\end{aligned}$$

*Proof.* From (41) we have

$$\varphi_1(E_{\Lambda_n}) = \text{Tr} \left[ \omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} \text{Tr}_{n-1} (K_{[n-1,n]} \mathbf{h}_n E_{\Lambda_n} K_{[n-1,n]}^*) \cdots K_{[0,1]}^* \right].$$

One can calculate that

$$\begin{aligned}\text{Tr}_{n-1} (K_{[n-1,n]} \mathbf{h}_n E_{\Lambda_n} K_{[n-1,n]}^*) &= \text{Tr}_{x_{W_{n-1}}^{(1)}} \left( A_{(x_{W_{n-1}}^{(1)}, x_{W_n}^{(1)}, x_{W_n}^{(2)})} (\mathbf{1}^{x_{W_{n-1}}^{(1)}} \otimes e_{1,1} h^{x_{W_n}^{(1)}} \otimes h^{x_{W_n}^{(2)}}) \right. \\ &\quad \left. A_{(x_{W_{n-1}}^{(1)}, x_{W_n}^{(1)}, x_{W_n}^{(2)})}^* \right) \otimes \bigotimes_{x \in W_{n-1} \setminus \{x_{W_n}^{(1)}\}} h^{(x)} \\ &= \frac{1}{2} \left[ (\xi_0 + \xi_3)(\tau_1\xi_0 + \tau_2\xi_3) \hat{\mathbf{h}}_{n-1} + \frac{\tau_3}{2}(\xi_0 + \xi_3)^2 \check{\mathbf{h}}_{n-1} \right].\end{aligned}$$

Hence,

$$\begin{aligned}\varphi_1(E_{\Lambda_n}) &= \frac{1}{2}(\xi_0 + \xi_3)(\tau_1\xi_0 + \tau_2\xi_3) \text{Tr} \left[ \omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} \hat{\mathbf{h}}_{n-1} K_{[n-2,n-1]}^* \cdots K_{[0,1]}^* \right] \\ &\quad + \tau_3 \left( \frac{\xi_0 + \xi_3}{2} \right)^2 \text{Tr} \left[ \omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} \check{\mathbf{h}}_{n-1} K_{[n-2,n-1]}^* \cdots K_{[0,1]}^* \right] \\ &= \frac{1}{2} \left[ (\xi_0 + \xi_3)(\tau_1\xi_0 + \tau_2\xi_3) \hat{\psi}_{n-1} + \frac{\tau_3}{2}(\xi_0 + \xi_3)^2 \check{\psi}_{n-1} \right].\end{aligned}$$

Now using the values of  $\hat{\psi}_{n-1}$  and  $\check{\psi}_{n-1}$  given by the previous lemma we obtain the result.  $\square$

Now we consider the state  $\varphi_2$ . Recall that this state is defined by  $\omega_0 = \frac{1}{\xi_0} \mathbf{1}$  and  $h^x = h' = \xi_0 \mathbf{1} - \xi_3 \sigma$ . Define two elements of  $\mathcal{B}_{W_n}$  by

$$\begin{aligned}\hat{\mathbf{h}}'_n &:= \mathbf{1}^{x_{W_n}^{(1)}} \otimes \bigotimes_{x \in W_n \setminus \{x_{W_n}^{(1)}\}} h'^{(x)} \\ \check{\mathbf{h}}'_n &:= \sigma^{x_{W_n}^{(1)}} \otimes \bigotimes_{x \in W_n \setminus \{x_{W_n}^{(1)}\}} h'^{(x)}\end{aligned}$$

Using the same argument like in the proof of Lemma 6.5 we can prove the following auxiliary fact.

**Lemma 6.7.** *Let*

$$\begin{aligned}\hat{\phi}_n &:= \text{Tr}_{n-1} [\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} \hat{\mathbf{h}}'_n K_{[n-1,n]}^* \cdots K_{[0,1]}^*] \\ \check{\phi}_n &:= \text{Tr}_{n-1} [\omega_0 K_{[0,1]} \cdots K_{[n-1,n]} \check{\mathbf{h}}'_n K_{[n-1,n]}^* \cdots K_{[0,1]}^*]\end{aligned}$$

*Then there are two pairs of reals  $(\hat{\pi}_1, \hat{\pi}_2)$  and  $(\check{\pi}_1, \check{\pi}_2)$  depending on  $\theta$  such that*

$$\begin{cases} \hat{\phi}_n = \hat{\pi}_1 + \hat{\pi}_2 \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^n, \\ \check{\phi}_n = \check{\pi}_1 + \check{\pi}_2 \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^n \end{cases}$$

where

$$\begin{aligned}\hat{\pi}_1 &= \frac{\tau_3^2}{3\tau_3 - 2\tau_1}, \quad \hat{\pi}_2 = \frac{2\tau_3(\tau_3 - \tau_1)}{3\tau_3 - 2\tau_2}, \\ \check{\pi}_1 &= -\frac{2\tau_2\tau_3^2\xi_3}{3\tau_3 - 2\tau_1}, \quad \check{\pi}_2 = \frac{2\tau_2\tau_3^2\xi_3}{3\tau_3 - 2\tau_1}.\end{aligned}$$

**Proposition 6.8.** *For each  $n \in \mathbb{N}$  one has*

$$\begin{aligned}\varphi_2(E_{\Lambda_n}) &= \frac{1}{2} \left[ (\xi_0 - \xi_3)(\xi_0\tau_1 - \xi_3\tau_2)\hat{\pi}_1 + \frac{\tau_3}{2}(\xi_0 - \xi_3)^2\check{\pi}_1 \right] \\ &\quad + \frac{1}{2} \left[ (\xi_0 - \xi_3)(\xi_0\tau_1 - \xi_3\tau_2)\hat{\pi}_2 + \frac{\tau_3}{2}(\xi_0 - \xi_3)^2\check{\pi}_2 \right] \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^{n-1}.\end{aligned}$$

*Proof.* From (42) we find

$$\varphi_2(E_{\Lambda_n}) = \text{Tr} \left[ \omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} \text{Tr}_{n-1} (K_{[n-1,n]} \mathbf{h}'_n E_{\Lambda_n} K_{[n-1,n]}^*) \cdots K_{[0,1]}^* \right]. \quad (45)$$

We easily calculate that

$$\text{Tr}_{n-1} (K_{[n-1,n]} h'_n a_{\Lambda_n} K_{[n-1,n]}^*) = \frac{1}{2} (\xi_0 - \xi_3) (\tau_1 \xi_0 - \tau_2 \xi_3) \hat{\mathbf{h}}'_{n-1} + \tau_3 \left( \frac{\xi_0 - \xi_3}{2} \right)^2 \check{\mathbf{h}}'_{n-1}.$$

Hence, from (45) one gets

$$\varphi_2(E_{\Lambda_n}) = \frac{1}{2} \left[ (\xi_0 - \xi_3) (\tau_1 \xi_0 - \tau_2 \xi_3) \hat{\phi}_{n-1} + \frac{\tau_3}{2} (\xi_0 - \xi_3)^2 \check{\phi}_{n-1} \right].$$

Using the values of  $\hat{\phi}_{n-1}$  and  $\check{\phi}_{n-1}$  given in Lemma 6.7, we obtain the desired assertion.  $\square$

**Theorem 6.9.** *The two QMC  $\varphi_1$  and  $\varphi_2$  are not quasi-equivalent.*

*Proof.* For any  $\forall n \in \mathbf{N}$  it is clear that  $E_{\Lambda_n} \in \mathcal{B}_{\Lambda_n} \setminus \mathcal{B}_{\Lambda_{n-1}}$ . Therefore, for any finite subset  $\Lambda \in L$ , there exists  $n_0 \in \mathbf{N}$  such that  $\Lambda \subset \Lambda_{n_0}$ . Then for all  $n > n_0$  one has  $E_{\Lambda_n} \in \mathcal{B}_{\Lambda_n} \setminus \mathcal{B}_{\Lambda}$ . It is clear that

$$\|E_{\Lambda_n}\| = \|e_{1,1}^{x_{W_n}^{(1)}} \bigotimes_{y \in L \setminus \{x_{W_n}^{(1)}\}} \mathbf{1}^y\| = \|e_{1,1}\| = \frac{1}{2}.$$

From Propositions 6.6 and 6.8 we obtain

$$\begin{aligned}|\varphi_1(E_{\Lambda_n}) - \varphi_1(E_{\Lambda_n})| &= \frac{1}{2} \left| \left[ (\xi_0 + \xi_3)(\xi_0\tau_1 + \xi_3\tau_2)\hat{\rho}_1 + \tau_3(\xi_0 + \xi_3)^2\check{\rho}_1 \right] \right. \\ &\quad - \left[ (\xi_0 - \xi_3)(\xi_0\tau_1 - \xi_3\tau_2)\hat{\pi}_1 + \tau_3(\xi_0 - \xi_3)^2\check{\pi}_1 \right] \\ &\quad + \left[ (\xi_0 + \xi_3)(\xi_0\tau_2 + \xi_3\tau_2)\hat{\rho}_2 + \tau_3(\xi_0 + \xi_3)^2\check{\rho}_2 \right] \\ &\quad \left. - \left[ (\xi_0 - \xi_3)(\xi_0\tau_2 - \xi_3\tau_2)\hat{\pi}_2 + \tau_3(\xi_0 - \xi_3)^2\check{\pi}_2 \right] \right| \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^{n-1} \Big| \\ &\geq I_1 - I_2 \left| \frac{\tau_1}{\tau_3} - \frac{1}{2} \right|^{n-1}\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2} \left| [(\xi_0 + \xi_3)(\xi_0\tau_1 + \xi_3\tau_2)\hat{\rho}_1 + \tau_3(\xi_0 + \xi_3)^2\check{\rho}_1] \right. \\
&\quad \left. - [(\xi_0 - \xi_3)(\xi_0\tau_1 - \xi_3\tau_2)\hat{\pi}_1 + \tau_3(\xi_0 - \xi_3)^2\check{\pi}_1] \right| \\
I_2 &= \frac{1}{2} \left| [(\xi_0 + \xi_3)(\xi_0\tau_2 + \xi_3\tau_2)\hat{\rho}_2 + \tau_3(\xi_0 + \xi_3)^2\check{\rho}_2] \right. \\
&\quad \left. - [(\xi_0 - \xi_3)(\xi_0\tau_2 - \xi_3\tau_2)\hat{\pi}_2 + \tau_3(\xi_0 - \xi_3)^2\check{\pi}_2] \right|.
\end{aligned}$$

Due to  $\beta > 0, \theta = \exp 2\beta > 1, \tau_1 > 0, \tau_3 > 0, \xi_0 > 0, \xi_3 > 0$ , one can find that

$$I_1 = \frac{\tau_3\xi_3(2\tau_2 + \tau_3)}{3\tau_3 - 2\tau_1} > 0.$$

Now keeping in mind  $0 < \tau_1 \leq \tau_3$  we have

$$\left| \frac{\tau_1}{\tau_3} - \frac{1}{2} \right| \leq \frac{1}{2}$$

which yields

$$I_2 \left| \frac{\tau_1}{\tau_3} - \frac{1}{2} \right|^{n-1} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Then there exists  $n_1 \in \mathbb{N}$  such that  $\forall n \geq n_0$  one has

$$I_2 \left| \frac{\tau_1}{\tau_3} - \frac{1}{2} \right|^n \leq \frac{\varepsilon_1}{2}.$$

Hence, for all  $n \geq n_1$  we obtain

$$|\varphi_1(E_{\Lambda_n}) - \varphi_1(E_{\Lambda_n})| \geq \frac{\varepsilon_1}{2} = \varepsilon_1 \|E_{\Lambda_n}\|.$$

This, according to Theorem 6.4, means that the states  $\phi_1$  and  $\phi_2$  are not quasi-equivalent. The proof is complete.  $\square$

Now Theorems 5.3, 6.2 and 6.9 imply Theorem 1.1.

## 7 Proof of Theorem 1.2

In this section we prove Theorem 1.2. In this section we assume that  $\Delta(\theta) > 0$  which ensures the existence of the state  $\varphi_1$  (see Theorem 5.3).

For each  $n \in \mathbb{N}$  let us define the following element:

$$a_\sigma^{\Lambda_n} = \sigma^{(W_{n+1}(1))}. \quad (46)$$

Clearly, one has  $a_\sigma^{\Lambda_n} \in \mathcal{B}_{\Lambda_n}$ .

**Proposition 7.1.** *Let  $\varphi_\alpha$  be the QMC associate to the pair  $(\omega_\alpha, h_\alpha)$ . Then one has*

$$\varphi_\alpha(a_\sigma^{\Lambda_n}) = 0.$$

*Proof.* According to Proposition 5.1 we have

$$\varphi_\alpha(a_\sigma^{\Lambda_n}) = \text{Tr} \omega_0 K_{[0,1]} \cdots K_{[n-2,n-1]} \text{Tr} r_{n-1} [K_{[n-1,n]} a_\sigma^{\Lambda_n} \mathbf{h}_{\alpha,n} K_{[n-1,n]}^*] K_{[n-2,n-1]}^* \cdots K_{[0,1]}^*. \quad (47)$$

One can calculate that

$$\begin{aligned} \text{Tr}_n [K_{[n-1,n]} a_\sigma^{\Lambda_n} K_{[n-1,n]}^*] &= \text{Tr}_{W_n(1)} [A_{(x,(x,1),(x,2))} (\sigma h_\alpha^{(x,1)} \otimes h_\alpha^{(x,2)}) A_{(x,(x,1),(x,2))}^*] \\ &\quad \otimes \bigotimes_{x \in W_n \setminus W_n(1)} \text{Tr}_x [A_{(x,(x,1),(x,2))} h_\alpha^{(x,1)} \otimes h_\alpha^{(x,2)}) A_{(x,(x,1),(x,2))}^*] \\ &= \frac{1}{2} \tau_3 \alpha \sigma^{W_{n-1}(1)} \mathbf{h}_{\alpha,n-1} \\ &= \frac{1}{2} \tau_3 \alpha a_\sigma^{\Lambda_{n-1}} \mathbf{h}_{\alpha,n-1} \end{aligned}$$

Therefore, from (47) with the last equality we obtain

$$\begin{aligned} \varphi_\alpha(a_\sigma^{\Lambda_n}) &= \frac{\tau_3 \alpha}{2} \varphi_\alpha(a_\sigma^{\Lambda_{n-1}}) \\ &\vdots \\ &= \left(\frac{\tau_3 \alpha}{2}\right)^n \varphi_{\alpha,0}(a_\sigma^{\Lambda_0}) \\ &= \left(\frac{\tau_3 \alpha}{2}\right)^n \text{Tr}(\omega_0 \sigma) = 0. \end{aligned}$$

This completes the proof.  $\square$

Now using the same argument as in the proof of Proposition 6.6 one can prove the following

**Proposition 7.2.** *For each  $n \in \mathbb{N}$  one has*

$$\varphi_1(a_\sigma^{\Lambda_{n+1}}) = \left[ (\tau_1 + \tau_2) \xi_0 \xi_3 \hat{\rho}_1 + \frac{\tau_3}{2} (\xi_0^2 + \xi_3^2) \check{\rho}_1 \right] + \left[ (\tau_1 + \tau_2) \xi_0 \xi_3 \hat{\rho}_2 + \frac{\tau_3}{2} (\xi_0^2 + \xi_3^2) \check{\rho}_2 \right] \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^n$$

Now we are ready to prove Theorem 1.2.

*Proof.* First note that the state  $\varphi_\alpha$  is a factor state, since it also satisfies the mixing property like the state  $\varphi_1$ .

Now from  $0 < \tau_1 < \tau_3$  one finds  $|\frac{\tau_1}{\tau_3} - 1| < \frac{1}{2}$  which yields

$$\left( \frac{\tau_1}{\tau_3} - 1 \right)^n \rightarrow 0.$$

Due to Propositions 7.1 and 7.2 we get

$$\begin{aligned} |\varphi_\alpha(a_\sigma^{\Lambda_{n+1}}) - \varphi_1(a_\sigma^{\Lambda_{n+1}})| &= |\varphi_1(a_\sigma^{\Lambda_{n+1}})| \\ &= |[(\tau_1 + \tau_2) \xi_0 \xi_3 \hat{\rho}_1 + \frac{1}{2} \tau_3 (\xi_0^2 + \xi_3^2) \check{\rho}_1] \\ &\quad + [(\tau_1 + \tau_2) \xi_0 \xi_3 \hat{\rho}_2 + \frac{1}{2} \tau_3 (\xi_0^2 + \xi_3^2) \check{\rho}_2] \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^n| \\ &\geq |(\tau_1 + \tau_2) \xi_0 \xi_3 \hat{\rho}_1 + \frac{1}{2} \tau_3 (\xi_0^2 + \xi_3^2) \check{\rho}_1| \\ &\quad - \left| (\tau_1 + \tau_2) \xi_0 \xi_3 \hat{\rho}_2 + \frac{1}{2} \tau_3 (\xi_0^2 + \xi_3^2) \check{\rho}_2 \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^n \right|, \quad (48) \end{aligned}$$

here  $\hat{\rho}_1, \hat{\rho}_2$  and  $\check{\rho}_1, \check{\rho}_2$  are defined by (43), (44), respectively.

We can check that  $\varepsilon_0 := [(\tau_1 + \tau_2)\xi_0\xi_3\hat{\rho}_1 + \frac{1}{2}\tau_3(\xi_0^2 + \xi_3^2)\check{\rho}_1] > 0$ . Therefore, we have

$$|(\tau_1 + \tau_2)\xi_0\xi_3\hat{\rho}_2 + \frac{1}{2}\tau_3(\xi_0^2 + \xi_3^2)\check{\rho}_2 \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^n| \rightarrow 0.$$

This means that there is  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  one gets

$$\left| (\tau_1 + \tau_2)\xi_0\xi_3\hat{\rho}_2 + \frac{1}{2}\tau_3(\xi_0^2 + \xi_3^2)\check{\rho}_2 \left( \frac{\tau_1}{\tau_3} - \frac{1}{2} \right)^n \right| \leq \varepsilon_0/2.$$

This due to (48) implies

$$|\varphi_\alpha(a_\sigma^{\Lambda_{n+1}}) - \varphi_1(a_\sigma^{\Lambda_{n+1}})| \geq \frac{\varepsilon_0}{2}$$

for all  $n \geq n_0$ .

For  $\varepsilon = \frac{\varepsilon_0}{2}$ , and  $\Lambda \subset_{fin} L$ , there exists  $n_1 \in \mathbb{N}$  such that  $\Lambda \subset \Lambda_{n_1}$

$$|\varphi_\alpha(a_\sigma^{\Lambda_{n+1}}) - \varphi_1(a_\sigma^{\Lambda_{n+1}})| \geq \varepsilon$$

for all  $n \geq \max\{n_0, n_1\}$ . This from Theorem 6.4 gets the desired statement.  $\square$

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